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The relative motion of the core and mantle of a planet in the gravitational field of a point mass☆

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Abstract

A model of a planet, consisting of two solid bodies – a core and a mantle – between which there is a spherical layer of a viscous incompressible liquid, is considered. The gravitational interaction between the core and the mantle is taken into account. The problem is investigated in a limited formulation, when the mass centre of the planet moves in a fixed elliptical orbit in the gravitational field of a point mass, while the mutual displacements of the core and the mantle are to be determined. The mutual displacements of the core and the mantle of the planet, and also the velocity field of the viscous liquid in the spherical layer, are obtained using multiparameter perturbation theory, where the Reynolds number, the orbit eccentricity and the ratio of the radius of the planet to the distance to its attracting centre are taken as small parameters. In addition, an approximate theory of gyroscopes is used to analyse the equations of motion. The results obtained are illustrated by the example of the motion of the Earth-Moon system.

The complex internal structure of the planets of the solar system produces relative motion of its component parts and affects the dynamic processes (the rotation of a planet around its mass centre, tidal phenomena, evolution of the orbit, and tectonic processes as a consequence of the relative displacements of parts of the planet).[1–4] The choice of any particular model to describe the motion of the planet and its component parts depends on its dimensions (mass), which determine the gravitational field inside the planet and affect the physical state of the planet material. If the planet dimensions are small, it represents a solid deformable body. When the mass of the planet increases a liquid layer can be formed inside it.

A model was considered previously in Ref. 5 under the conditions that the continuous medium between the core and the mantle of the planet has viscoelastic properties, but was unable to investigate "rotations" of the core with respect to the mantle.

1. Formulation of the problem

Suppose the planet consists of a core and a mantle, between which there is a spherical layer of uniform viscous incompressible liquid. This type of model corresponds to modern representations of the internal structure of the Earth, which consists of a solid core, a spherical layer of viscous liquid and a solid mantle. Phase transitions from one type

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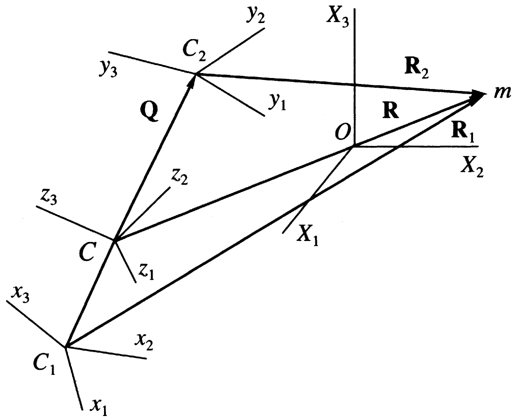

Fig. 1.

of continuous medium to the other (solid – liquid – solid) are related to the value of the pressure inside the Earth, which increases as one approaches the centre of the Earth, and the temperature distribution inside the Earth. Inside the mantle the temperature and pressure are low and, as a consequence of this, the material is in a solid state (the model of an absolutely rigid or deformable body). As one approaches the centre of the Earth, the temperature and pressure increase, and at a certain distance r from the centre of the Earth ($r = b$) the pressure $p(r)$ becomes equal to $p_*(T(b))$, where $p_*(T)$ is a function which defines the relation between the pressure and the temperature T, at which the solid - liquid phase transition occurs. Further, as one approaches the centre of the Earth, the pressure and temperature vary in such a way that when $r = a$ the pressure becomes equal to $p_*(T(a))$, at which the inverse transition from the liquid phase to the solid phase occurs. The dependence of the pressure on the distance from the centre of the Earth r is mainly determined by the gravitational interaction between the particles inside the Earth and the centrifugal forces due to the daily rotation of the Earth, which does not exceed 0.3% of the gravitational forces. In view of this the isobars inside the Earth and the boundaries of the liquid layer can be assumed to be spherical. Information on the internal structure of the Earth is based on an investigation of the way in which seismic waves travel through it (Fig. 1).[2,3]

Suppose the core and the mantle of the planet are solid bodies. The system of coordinates $C_1x_1x_2x_3$ is connected with the core, which occupies a region $V_1 = \{\mathbf{r}_1^2 < a^2, \mathbf{r}_1 = (x_1, x_2, x_3)\}$, the point C_1 is the mass centre of the core and $J_1 = \mathrm{diag}\{A_1, A_1, C_1\}$ is the inertia tensor of the core with respect to the system of coordinates $C_1x_1x_2x_3$. We will assume that the principal central moment of inertia $C_1 > A_1$ (this is related to the fairly rapid rotation of the planet about the C_1x_3 axis). As regards the mantle, we will also assume that it occupies a region V_2, the inner surface of which is given by the equation $\mathbf{r}_2^2 = b^2$, $\mathbf{r}_2 = (y_1, y_2, y_3)$ in the system of coordinates $C_2y_1y_2y_3$. Here C_2 is the mass centre of the mantle and $b - a = l$ is the thickness of the liquid layer between the core and the mantle. The inertia tensor of the mantle in the system of coordinates $C_2y_1y_2y_3$ is equal to $J_2 = \mathrm{diag}\{A_2, A_2, C_2\}$, $C_2 > A_2$. The velocity field $\mathbf{v}(\mathbf{r}, t)$, $a \le |\mathbf{r}| \le b$ of the liquid will be considered on the assumption that the systems of coordinates $C_1x_1x_2x_3$ and $C_2y_1y_2y_3$ coincide, while the velocity field describing their mutual motion is not identically equal to zero and determines the boundary values of the velocity field of points of the liquid layer.

The system of coordinates $CX_1X_2X_3$ is connected with the mass centre of the planet, and its axes are parallel to the axes of the inertial system of coordinates $OX_1X_2X_3$, the origin of which is the mass centre of the planet and of the point mass of mass m (see the figure). When the mass centre of the mantle is displaced with respect to the mass centre of the core, the following vector equalities hold in the system of coordinates $CX_2X_2X_3$

$$\begin{gathered} CC_1 = \xi_1\mathbf{Q}, \; CC_2 = \xi_2\mathbf{Q}; \; \xi_1 = -\frac{m_r}{m_{10}}, \; \xi_2 = \frac{m_r}{m_{20}}, \; m_r = \frac{m_{10}m_{20}}{M}, \; M = m_0 + m_1 + m_2 \\ m_{10} = m_1 - m_{01} > 0, \; m_{20} = m_2 + m_{02}, \; m_0 = m_{02} - m_{01}, \; m_{01} = \frac{4}{3}\pi\rho a^3, \; m_{02} = \frac{4}{3}\pi\rho b^3 \end{gathered} \tag{1.1}$$

where m_0, m_1 and m_2 are the masses of the liquid layer, the core and the mantle respectively, ρ is the density of the liquid and $\mathbf{Q}$ is the vector C_1C_2.

We further obtain $\mathbf{P}_k = \xi_k\mathbf{Q} + \Gamma_k\mathbf{r}_k$, $\mathbf{r}_k \in V_k$ $(k = 1, 2)$, where $\mathbf{P}_1$ and $\mathbf{P}_2$ are the vectors of points of the core and the mantle, Γ_1 and Γ_2 are orthogonal operators of transition from the systems of coordinates $C_1x_1x_2x_3$, $C_2y_1y_2y_3$ to the system $CX_1X_2X_3$. If $\mathbf{Q} = 0$ and $\Gamma_1 = \Gamma_2 = \Gamma_0$ the core of the planet is inside the mantle and their principal axes of inertia coincide, while the liquid fills the spherical layer between the core and the mantle. The rotation operator Γ_0 determines the transition from the system of coorindates $CX_1X_2X_3$ to the system $Cz_1z_2z_3$, connected with the planet in unperturbed motion.

When the planet moves, there are small displacements and rotations of the mantle of the planet with respect to its core, with the exception of rotations about the axes of dynamic symmetry of the core and the mantle, which may have a secular form. In the system of coordinates $Cz_1z_2z_3$ this displacement is equal to $\mathbf{q} = \Gamma_2^1\mathbf{Q}$, while the relative rotation operator $\Gamma = \Gamma_1^{-1}\Gamma_2$ is the matrix of rotation of the mantle with respect to the system of coordinates $C_1x_1x_2x_3$, connected with the core. The displacement vector $\mathbf{q}$ is small in the sense that the ratio $|\mathbf{q}|/l$ is small. If the operators Γ_1 and Γ_2 are specified using the Euler angles, the following equalities will hold

$$\Gamma_k = O_3(\psi + \psi_k)O_1(\theta + \theta_k)O_3(\varphi + \varphi_k), \quad k = 1, 2$$

$$O_1(\chi) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\chi & -\sin\chi \\ 0 & \sin\chi & \cos\chi \end{Vmatrix}, \quad O_3(\chi) = \begin{Vmatrix} \cos\chi & -\sin\chi & 0 \\ \sin\chi & \cos\chi & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{1.2}$$

The Euler angles ψ, θ, φ define the rotation operator Γ_0 and describe the rotation of the planet, when its core and mantle are a single whole, and the perturbations ψ_k, θ_k, φ_k $(k = 1, 2)$ are equal to zero. Using relations (1.2), we obtain

$$(\Gamma_1\mathbf{e}_3, \Gamma_2\mathbf{e}_3) \cong 1 - \frac{1}{2}[(\Delta\theta)^2 + (\Delta\psi)^2\sin^2\theta], \quad \Delta\theta = \theta_2 - \theta_1, \quad \Delta\psi = \psi_2 - \psi_1 \tag{1.3}$$

where the expression in square brackets is the square of the angle between the axes of symmetry of the core C_1x_3 and the mantle C_2y_3. In (1.3) we have omitted terms of the third order of smallness and higher in the small quantities $\Delta\psi$, $\Delta\theta$. Unlike the case considered earlier[5] the perturbation $\Delta\varphi = \varphi_2 - \varphi_1$ may have a secular character, since the core and the mantle are not connected elastically, and the gravitational interaction in the system is independent of $\Delta\varphi$.

As the unperturbed motion of the system we will consider the motion of the planet as a rigid body, when there are no relative displacements of the core, the mantle and the liquid. We will also assume that the mass centre of the planet describes an elliptic orbit under the action of the gravitation of a point mass m and that the planet rotates with constant velocity about the axis of symmetry. In the limited formulation of the problem, we will take the motion of the mass centre of the planet as unperturbed and we will investigate the mutual motion of the core, the mantle and the liquid.

The kinetic energy of the system can be calculated using Koenig's theorem and is equal to

$$T = \frac{mM}{2(m+M)}\dot{\mathbf{R}}^2 + \frac{m_1\xi_1^2 + m_2\xi_2^2}{2}(\dot{\mathbf{q}} + [\boldsymbol{\omega}\times\mathbf{q}])^2 + \frac{1}{2}\sum_{k=1}^{2}(J_k\boldsymbol{\omega}_k, \boldsymbol{\omega}_k) + \frac{\rho}{2}\int_{V_0}(\mathbf{v} + [\boldsymbol{\omega}\times\mathbf{r}])^2 dx$$

where $\mathbf{R}$ is the vector connecting the mass centre of the planet and the point with mass m, $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ are the angular velocities of the core and the mantle in systems of coordinates connected with each of these, and $\mathbf{v}(\mathbf{r}, t)$ is the velocity field of the liquid in region V_0 between the core and the mantle, specified in a uniformly rotating system of coordinates $Cz_1z_2z_3$. The following relations hold

$$\frac{1}{2}(J_k\boldsymbol{\omega}_k, \boldsymbol{\omega}_k) = \frac{1}{2}A_k[\dot{\psi}_k^2\sin^2(\theta + \theta_k) + \dot{\theta}_k^2] + \frac{1}{2}C_k[\dot{\varphi} + \dot{\varphi}_k + \dot{\psi}_k\cos(\theta + \theta_k)]^2, \quad k = 1, 2$$

In the Stokes approximation (we will assume that the Reynolds numbers $\mathrm{Re} = \nu^{-1}l\max(|\dot{\mathbf{q}}|, |\Gamma_2\boldsymbol{\omega}_2 - \Gamma_1\boldsymbol{\omega}_1|l)$ are small) and assuming the flow to be quasi-stationary, the equations of motion of the viscous incompressible liquid in the liquid layer in the uniformly rotating system of coordinates $Cz_1z_2z_3$ and the boundary conditions on the surface of the core ∂V_1 and on the inner surface of the mantle ∂V_2 have the form[6]

$$\rho^{-1}\nabla p = -\nabla\Pi(\mathbf{r},t) + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0, \quad \Pi(\mathbf{r},t) = \Pi_g(\mathbf{r},t) - [\boldsymbol{\omega}\times\mathbf{r}]^2/2$$
$$\mathbf{v}(\mathbf{r},t)|_{\mathbf{r}\in\partial V_k} = \xi_k\dot{\mathbf{q}} + [(\Gamma_0^{-1}\Gamma_k\boldsymbol{\omega}_k - \boldsymbol{\omega})\times\mathbf{r}], \quad k = 1,2 \tag{1.4}$$

Here ν is the coefficient of kinematic viscosity of the liquid, $p(\mathbf{r}, t)$ is the pressure of the liquid and $\Pi_g(\mathbf{r}, t)$ is the potential of the mass forces of gravitational interaction of all parts of the system. Henceforth, when determining the velocity field we will neglect small displacements of the mantle with respect to the core (the vector $\mathbf{q}$) and we will assume that Eq. (1.4) hold in the spherical layer $V_0 = \{\mathbf{r}: a \le |\mathbf{r}| \le b\}$ and on its boundaries respectively.

The equations of motion and continuity (1.4) are linear in the unknown functions $p(\mathbf{r}, t)$ $\mathbf{v}(\mathbf{r}, t)$, which enables us to obtain their solution in the form

$$p = p_0 + p_1 + p_2, \quad \mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \quad p_0 = -\rho\Pi$$

where the functions introduced satisfy the following equations and boundary conditions

$$\rho^{-1}\nabla p_k = \nu\Delta\mathbf{v}_k, \quad \operatorname{div}\mathbf{v}_k = 0; \quad k = 1,2 \tag{1.5}$$

$$\mathbf{v}_1(\mathbf{r},t)|_{|\mathbf{r}|=a} = \xi_1\dot{\mathbf{q}}, \quad \mathbf{v}_1(\mathbf{r},t)|_{|\mathbf{r}|=b} = \xi_2\dot{\mathbf{q}} \tag{1.6}$$

$$\mathbf{v}_2(\mathbf{r},t)|_{|\mathbf{r}|=a} = [(\Gamma_0^{-1}\Gamma_1\boldsymbol{\omega}_1 - \boldsymbol{\omega})\times\mathbf{r}], \quad \mathbf{v}_2(\mathbf{r},t)|_{|\mathbf{r}|=b} = [(\Gamma_0^{-1}\Gamma_2\boldsymbol{\omega}_2 - \boldsymbol{\omega})\times\mathbf{r}] \tag{1.7}$$

Bearing in mind the spherical symmetry of the region occupied by the liquid, it is more convenient to solve problem (1.5), (1.6) in a spherical system of coordinates, assuming that the velocity of the mantle with respect to the core $\dot{\mathbf{q}}$ is directed along the Cz axis. Subtracting the velocity of translational motion of the core from the velocity field of the liquid, we obtain problem (1.5), (1.6) with the changed boundary conditions

$$\upsilon_r(a,\theta) = \upsilon_\theta(a,\theta) = 0, \quad \upsilon_r(b,\theta) = \dot{q}\cos\theta, \quad \upsilon_\theta(b,\theta) = -\dot{q}\sin\theta; \quad \dot{q} = |\dot{\mathbf{q}}|$$

Here $\nu_r(r, \theta)$, $\nu_\theta(r, \theta)$ are the components of the velocity field of the liquid in a spherical system of coordinates, while the component of the velocity field corresponding to the angle φ of the spherical system of coordinates is identically equal to zero.

By analogy with the Stokes solution, we will obtain the solution of this problem in the form

$$\upsilon_r(r,\theta) = f(r)\cos\theta, \quad \upsilon_\theta(r,\theta) = -g(r)\sin\theta, \quad p(r,\theta) = p\nu h(r)\cos\theta$$
$$f = Ar^{-3} + Br^{-1} + C + Dr^2, \quad g(r) = -2Ar^{-3} + 2Br^{-1} + C + 2Dr^2, \quad h = Br^{-2} + 10Dr$$

The coefficients A, B, C and D are found from the boundary conditions

$$A = 3a^3(\xi^4 + \xi^5)F(\xi)\dot{q}, \quad B = -a(4\xi + 4\xi^2 + 4\xi^3 + 9\xi^4 + 9\xi^5)F(\xi)\dot{q}$$

$$C = 6(\xi + \xi^2 + \xi^3 + \xi^4 + \xi^5)F(\xi)\dot{q}, \quad D = 2a^{-2}(\xi + \xi^2 + \xi^3)F(\xi)\dot{q}$$

$$F(\xi) = (1-\xi)^{-3}(4 + 7\xi + 4\xi^2)^{-1}, \quad \xi = b/a > 1$$

The stresses on elements of the surface of the spherical core, taking the boundary conditions and the incompressibility condition into account, are[6]

$$p_{rr} = -p + 2\mu\frac{\partial\upsilon_r}{\partial r} = -\mu h(a)\cos\theta, \quad p_{r\theta} = \mu\left(\frac{\partial\upsilon_\theta}{\partial r} + a^{-1}\frac{\partial\upsilon_r}{\partial\theta}\right) = -\mu g'(a)\sin\theta$$

Here $\mu = \rho\nu$ and the prime denotes differentiation with respect to r.

The resulting pressure on the core is directed along the vector $\dot{\mathbf{q}}$ and is calculated from the formula

$$F_1 = \int_{\partial V_1}\int(p_{rr}\cos\theta - p_{r\theta}\sin\theta)d\sigma = -2\pi\mu a^2\dot{q}\int_0^{\pi}[h(a)\cos^2\theta - g'(a)\sin^2\theta]\sin\theta d\theta =$$

$$= -\frac{4}{3}\pi\mu a^2\dot{q}[h(a) - 2g'(a)] = -d_1\dot{q}, \quad d_1 = \frac{4}{3}\pi\mu a\frac{4\xi + 4\xi^2 + 4\xi^3 + 21\xi^4 + 21\xi^5}{(\xi - 1)^3(4 + 7\xi + 4\xi^2)}$$

The force acting on the inner surface of the mantle, is equal in value and opposite in direction to the force F_1, while the corresponding dissipative function has the form

$$D_1 = \frac{1}{2}d_1\dot{\mathbf{q}}^2 \tag{1.8}$$

Problem (1.5), (1.7) is also solved in spherical coordinates.[6] As a result we determine the moment of interaction of the core and the mantle due to their mutual rotation and the corresponding dissipative function

$$M = d_2|\Gamma_1\boldsymbol{\omega}_1 - \Gamma_2\boldsymbol{\omega}_2|, \quad D_2 = \frac{1}{2}d_2(\Gamma_1\boldsymbol{\omega}_1 - \Gamma_2\boldsymbol{\omega}_2)^2; \quad d_2 = \frac{8\mu\pi a^3b^3}{b^3 - a^3} \tag{1.9}$$

It remains to consider the potential component of the pressure

$$p_0 = -\rho^{-1}\Pi(\mathbf{r}, t)$$

in the liquid layer and the action of this pressure on the mantle and on the core of the planet. We will consider the work due to virtual displacements of the field of potential forces $-\nabla\Pi(\mathbf{r}, t)$, $\mathbf{r} \in V_0$, where V_0 is the region occupied by the liquid. Using Gauss' formula and the condition of incompressibility of the liquid, we can represent this work in the form

$$\delta A_p = -\int_{V_0}\nabla\Pi\delta\mathbf{R}dx = -\int_{V_0}\operatorname{div}(\Pi\delta\mathbf{R})dx = -\int_{\partial V_0}\Pi\mathbf{n}\delta\mathbf{R}d\sigma \tag{1.10}$$

Here $\delta\mathbf{R}(\mathbf{r})$ is the field of virtual displacements which satisfy the relations div $\delta\mathbf{R} = 0$, and $\mathbf{n}(\mathbf{r}, t)$ is the outward normal to the surface of the liquid layer.[7]

Note that on the boundary of the liquid layer the virtual displacements are identical with the virtual displacements of the corresponding points of the surface of the core or of the inner surface of the mantle. Using this fact, we will split the last integral in (1.10) into two integrals. We apply Gauss' formula to each of these, extending the field of virtual displacements on the surface of the core into the region occupied by the core, like the field of virtual displacements of the core, and we extend the field of virtual displacements, specified on the inner surface of the mantle inside the whole core like the field of virtual displacements of the mantle, extended over the whole inner surface. As a result we obtain

$$\delta A_p = \int_{\partial V_1}\Pi\mathbf{n}_1\delta\mathbf{R}_1 d\sigma - \int_{\partial V_2}\Pi\mathbf{n}_2\delta\mathbf{R}_2 d\sigma = \int_{V_1}\nabla\Pi\delta\mathbf{R}_1 dx - \int_{V_1\cup V_0}\nabla\Pi\delta\mathbf{R}_2 dx$$

where $\mathbf{n}_1(\mathbf{r}, t)$ is the outward normal to the surface of the core, $\delta\mathbf{R}_1(\mathbf{r})$ is the field of virtual displacements of points of the core as a solid body, $\mathbf{n}_2(\mathbf{r}, t)$ is the outward normal to the inner surface of the mantle, directed towards the mantle, and $\delta\mathbf{R}_2(\mathbf{r})$ is the field of virtual displacements of points inside the whole cavity of the mantle, which is identical with the analogous field of a solid body, rigidly connected to the mantle. The property obtained holds for any form of the cavity filled with an incompressible liquid.

We will consider the motion of the core and the mantle of the planet with respect to an inertial system of coordinates $OX_1X_2X_3$, using Lagrange's equations of the second kind. We will take as the generalized coordinates the coordinates of the vector $\mathbf{q}$ and the perturbations of the Euler angles ψ_k, θ_k, φ_k ($k = 1,2$), while the radius vector $\mathbf{R}$, connecting the mass centre of the planet and the point mass m, corresponds to the unperturbed motion (see the figure).

As was shown above, the gravitational interaction of the core, mantle and liquid remains unchanged if we assume that the whole cavity $V_1 \cup V_0$ (the sphere inside the mantle) is filled with an incompressible liquid of density ρ, while the density of the material in the spherical region, occupied by the core, is reduced by the value of the density of the liquid ρ. The assumption that the corresponding regions are spherical is unimportant. From the point of view of gravitational interaction of the parts of the planet, and also of its gravitational interaction with external bodies, the planet can be assumed to consist of two solid parts: a mantle with an additional inner spherical cavity, represented by a solid body of constant density ρ, and a core, the density of the material of which is reduced by an amount ρ. The interaction between the core and the mantle is made up of their gravitational interaction and the interaction related to the presence of a layer of uniform viscous liquid between them. The gravitational interaction potential has the form[5]

$$\Pi_{12} = -\frac{g_1}{2}[q_1^2+q_2^2-2q_3^2]+\frac{h}{2}[(\Delta\psi)^2\sin^2\theta+(\Delta\theta)^2]$$

$$g_1 = \frac{\gamma m_{10}(C_2-A_2)}{2r_0^5}>0, \quad h = \frac{\gamma(C_1-A_1)(C_2-A_2)}{ar_0(r_0-a)^3}>0$$

where r_0 is the outer radius of the mantle (the radius of the planet) and γ is the universal gravitational constant.

We will determine the action on the core and on the mantle of the planet of the component of the pressure p_0, related to the rotation of the system of coordinates $Cz_1z_2z_3$ and equal to $\rho[\boldsymbol{\omega}\times\mathbf{r}]^2/2$. By changing from surface integrals to volume integrals we can obtain an expression for the equivalent potential of the centrifugal forces

$$\Pi_0 = \frac{m_{01}\xi_1^2-m_{02}\xi_2^2}{2}[\boldsymbol{\omega}\times\mathbf{q}]^2+\frac{m_{02}b^2-m_{01}a^2}{5}\boldsymbol{\omega}^2$$

in which we have omitted the last term, which is independent of the generalized coordinates. Taking the last expression for the potential of the centrifugal forces into account, we can represent the difference in the kinetic energy of the core-mantle system and the potential of the centrifugal forces in the form

$$T_{12}-\Pi_0 = \frac{m_1\xi_1^2+m_2\xi_2^2}{2}(\dot{\mathbf{q}}^2+2\dot{\mathbf{q}}[\boldsymbol{\omega}\times\mathbf{q}])+\frac{m_r}{2}[\boldsymbol{\omega}\times\mathbf{q}]^2+\frac{1}{2}\sum_{k=1}^{2}(J_k\boldsymbol{\omega}_k,\boldsymbol{\omega}_k)$$

We will obtain the potential energy of the gravitational interaction of the three components of the planet with the point mass m. The presence of a liquid layer between the core and the mantle can be taken into account by adding to the mantle a body of mass m_{02} with constant density completely filling the region $V_0\cup V_1$. At the same time, one must reduce the density of the material comprising the core of the planet by the value of the liquid density ρ. Hence, the gravitational interaction of the planet with the external point mass reduces to the interaction with this mass of two solids – the modified mantle and the modified core. By relations (1.1), the potential energy of this interaction can be represented by the functions[8]

$$\Pi_{k3}\cong-\gamma m\left[\frac{m_{k0}}{|\mathbf{R}_k|}-\frac{A_k-C_k}{2|\mathbf{R}_k|^3}(1-3\alpha_{3k}^2)\right]$$
$$\mathbf{R}_k = \mathbf{R}-\xi_k\Gamma_0\mathbf{q}, \quad \alpha_{3k} = \Gamma_k\mathbf{e}_3\mathbf{R}_k^\circ, \quad \mathbf{R}_k^\circ = \mathbf{R}_k/|\mathbf{R}_k|; \quad k = 1, 2 \tag{1.11}$$

In relations (1.11) there are terms, the order of smallness of which is determined by the power of the ratio r_0/R, and small terms of the order of $|\mathbf{q}|/R$, $\Delta\psi$, $\Delta\theta$ and higher. Taking this into account, we retain in expression (1.11) terms of the lowest order of smallness and we obtain

$$\Pi_{13}+\Pi_{23}\cong-\frac{\gamma mM}{R}+\frac{\gamma mm_r}{2R^3}[q^2-3(\mathbf{R}^\circ,\Gamma_0\mathbf{q})^2]+\Sigma$$

$$\Sigma = \frac{\gamma m}{2R^3}\sum_{k=1}^{2}(A_k-C_k)\Big\{1-3(\mathbf{R}^\circ,\Gamma_k\mathbf{e}_3)^2+$$

$$+3\frac{\xi_k}{R}[(\mathbf{R}^\circ,\Gamma_0\mathbf{q})+2(\mathbf{R}^\circ,\Gamma_k\mathbf{e}_3)(\Gamma_k\mathbf{e}_3,\Gamma_0\mathbf{q})-5(\mathbf{R}^\circ,\Gamma_k\mathbf{e}_3)^2(\mathbf{R}^\circ,\Gamma_0\mathbf{q})]\Big\}; \quad \mathbf{R}^\circ = \frac{\mathbf{R}}{|\mathbf{R}|}$$

In this limited formulation of the problem, the mass centre of the planet moves in an unperturbed Kepler orbit, while the core and the mantle of the planet constitute a single whole – a rigid body which rotates with constant velocity about the axis of symmetry. Hence, the unperturbed motion of the system is described by the relations

$$R = \frac{p}{1+e\cos\vartheta}, \quad \mathbf{R}^\circ = (\cos\vartheta, \sin\vartheta, 0)$$
$$\psi = \text{const}, \quad \theta = \text{const}, \quad \dot\varphi = \text{const}, \quad \psi_k = \theta_k = \dot\varphi_k = 0; \quad k = 1, 2$$

Here p, e and ϑ are a parameter, the eccentricity and true anomaly of the orbit of the mass centre of the planet. The Lagrange function is represented in the form

$$\begin{aligned} L &= \frac{m_{12}}{2}(\dot{\mathbf{q}}^2 + 2\dot{\mathbf{q}}[\boldsymbol\omega\times\mathbf{q}]) + \frac{m_r}{2}[\boldsymbol\omega\times\mathbf{q}]^2 + \\ &+ \frac{1}{2}\sum_{k=1}^{2}\{A_k[\dot\psi_k^2\sin^2(\theta+\theta_k) + \dot\theta_k^2] + C_k[\dot\varphi + \dot\varphi_k + \dot\psi_k\cos(\theta+\theta_k)]^2\} + \\ &+ \frac{g_1}{2}[q_1^2 + q_2^2 - 2q_3^2] - \frac{h}{2}[(\Delta\psi)^2\sin^2\theta + (\Delta\theta)^2] - \\ &- \frac{\gamma m m_r}{2R^3}[\mathbf{q}^2 - 3(\mathbf{R}^\circ, \Gamma_0\mathbf{q})^2] + \Sigma, \quad m_{12} = m_1\xi_1^2 + m_2\xi_2^2 \end{aligned} \tag{1.12}$$

We will use the Lagrange function (1.12) to set up the equations of motion of the mechanical system.

2. The equations of motion and investigation of their solutions

Lagrange's equation in the variable $\mathbf{q}$, taking the dissipative function (1.8) into account, has the form

$$\begin{aligned} &m_{12}\ddot{\mathbf{q}} + d_1\dot{\mathbf{q}} + 2m_{12}[\boldsymbol\omega\times\dot{\mathbf{q}}] + m_r[\boldsymbol\omega\times[\boldsymbol\omega\times\mathbf{q}]] + K_{12}\mathbf{q} + \frac{\gamma m m_r}{R^3}[\mathbf{q} - 3(\mathbf{R}^\circ, \Gamma_0\mathbf{q})\Gamma_0^{-1}\mathbf{R}^\circ] = \\ &= \frac{3\gamma m}{2R^4}\sum_{k=1}^{2}(A_k - C_k)[\Gamma_0^{-1}\mathbf{R}^\circ + 2(\mathbf{R}^\circ, \Gamma_k\mathbf{e}_3)\Gamma_0^{-1}\Gamma_k\mathbf{e}_3 - 5(\mathbf{R}^\circ, \Gamma_k\mathbf{e}_3)^2\Gamma_0^{-1}\mathbf{R}^\circ] \\ &K_{12} = g_1\operatorname{diag}\{-1, -1, 1/2\} \end{aligned} \tag{2.1}$$

In Eq. (2.1) we have retained terms that are linear in the variables $\mathbf{q}$, ψ_k, θ_k ($k=1, 2$). If the trivial solution of homogeneous equation (2.1) is asymptotically stable, its particular solution will describe forced oscillations about the position of relative equilibrium. Since the eccentricities of the orbits of planets of the Solar System are fairly small, it is of some interest initially to consider circular orbits, when $e=0$ and $p=R$. In Eq. (2.1) there are two periodic functions of time, corresponding to the frequency of natural rotation of the planet ω and the frequency of orbital motion $\dot\vartheta = \Omega = \sqrt{\gamma m R^{-3}}$. We will assume that $\Omega \ll \omega$ and average all the coefficients of the variables $\mathbf{q}$, ψ_k, θ_k ($k=1, 2$) over the variables φ and ϑ, and we will average the perturbing forces solely over the variable φ. As a result, Eq. (2.1) takes the form

$$\begin{aligned} &m_{12}\ddot{\mathbf{q}} + d_1\dot{\mathbf{q}} + 2m_{12}\omega[\mathbf{e}_3\times\dot{\mathbf{q}}] - m_r\omega^2[\mathbf{q} - (\mathbf{q}, \mathbf{e}_3)\mathbf{e}_3] + K_{12}\mathbf{q} + K_3\mathbf{q} = \\ &= -\frac{1}{R}\sum_{k=1}^{2}\xi_k M_k(-\sin\theta\sin g + 2\sin2\theta\sin g + 5\sin^3\theta\sin^3 g)\mathbf{e}_3 \\ &M_k = \frac{3}{2}\Omega^2(C_k - A_k), \quad g = \vartheta - \psi, \quad K_3 = \frac{1}{4}m_r\Omega^2(3\cos^2\theta - 1)\operatorname{diag}\{1, 1, -2\} \end{aligned} \tag{2.2}$$

Eq. (2.2) is equivalent to a system of three second-order differential equations, which can be split into a system of equations in the variables q_1, and q_2 and an equation in the variable q_3. Moreover, Eq. (2.2) do not contain the variables

ψ_k, θ_k ($k = 1, 2$). The equations in the variables q_1 and q_2 can be represented in the form

$$\ddot{q}_1 + d_{10}\dot{q}_1 - 2\omega\dot{q}_2 + k_{12}q_1 = 0, \quad \ddot{q}_2 + d_{10}\dot{q}_2 + 2\omega\dot{q}_1 + k_{12}q_2 = 0 \tag{2.3}$$

where

$$d_{10} = \frac{d_1}{m_{12}}, \quad k_{12} = -\frac{1}{m_{12}}\left[m_r\omega^2 + g_1 - \frac{1}{4}m_r\Omega^2(3\cos^2\theta - 1)\right] \tag{2.4}$$

Eq. (2.2) can be represented in projections onto the Cz_3 axis in the form

$$\begin{gathered}\ddot{q}_3 + d_{10}\dot{q}_3 + k_3q_3 = P_1\sin g + P_3\sin 3g \\ P_1 = P\left(2\sin 2\theta - \sin\theta + \frac{15}{4}\sin^3\theta\right), \quad P_3 = -\frac{5}{4}P\sin^3\theta \\ P = \frac{m_r}{Rm_{12}}\left(\frac{M_1}{m_{10}} - \frac{M_2}{m_{20}}\right), \quad k_3 = \frac{2g_1}{m_{12}} + \frac{m_r\Omega^2}{m_{12}}\left(\frac{3}{2}\sin^2\theta - 1\right)\end{gathered} \tag{2.5}$$

The system of Eq. (2.3) can be written more conveniently in the form of a single equation in the complex variable $z = q_1 + iq_2$. As a result we obtain a homogeneous linear equation with constant coefficients

$$\ddot{z} + 2\varepsilon\dot{z} + k_{12}z = 0, \quad \varepsilon = d_{10}/2 + i\omega \tag{2.6}$$

the general solution of which has the form

$$z = E_+\exp(\lambda_+t) + E_-\exp(\lambda_-t), \quad \lambda_\pm = -\varepsilon \pm \sqrt{\varepsilon^2 - k_{12}}$$

where E_+ and E_- are arbitrary complex constants.

According to the second formula of (2.4), the coefficient $k_{12} < 0$ and $\mathrm{Re}\lambda_+ > 0$, $\mathrm{Re}\lambda_- < 0$. Consequently, the trivial solution of Eq. (2.6) is unstable: the solution corresponding to the root of the characteristic equation λ_- attenuates, while that corresponding to λ_+ increases. As a result, the projection of the centre of the core onto the equatorial plane (the plane Cz_1z_2) describes an uncoiling spiral. This motion can be described by the formula

$$z(t) = r\exp(t\mathrm{Re}\lambda_+)\exp[i(t\mathrm{Im}\lambda_+ + \alpha)], \quad r\exp(i\alpha) = E_+$$

It follows from physical considerations that as the distance of the centres of the core and the mantle from the centre of the planet increases in the unperturbed motion of the point C, parts of the core and the mantle fall within the zone where the pressure is insufficient to maintain the phase states. This means that, on the part of the core surface receding from the centre of the planet, due to the reduction in the pressure, the solid phase will transfer into the liquid phase, and on the opposite side of the core, the reverse will occur, namely, the liquid phase will convert into the solid phase. A similar pattern will be observed on the inner surface of the mantle: in regions where the mantle approaches the core, the material will convert from the solid phase into the liquid phase, while on the opposite side, the liquid phase will convert into the solid phase. The modulus of the number $z(t)$ will not increase to infinity and in steady motion it will be constant. In this case the projection of the vector $\mathbf{q}$ onto the equatorial plane of the planet will rotate with a period of $2\pi|\mathrm{Im}\lambda_+|^{-1/2}$. The value of the displacement of the core with respect to the mantle in the equatorial plane depends on the physical properties of the planet material and on the pressure distribution inside the planet.

Natural oscillations of the core with respect to the mantle in the projection onto the axis of symmetry Cz_3 will be attenuated, since in Eq. (2.5) the quantities d_{10} and k_3 are positive. As a result, after a certain time only forced oscillations will be observed, described by the formula

$$q_3(t) = \sum_{n=1,3} P_n\frac{(k_3 - n^2\Omega^2)\sin ng - d_{10}n\Omega\cos ng}{(k_3 - n^2\Omega^2)^2 + (d_{10}n\Omega)^2}$$

Lagrange's equations, corresponding to the angular variables ψ_k, θ_k, φ_k, ($k=1, 2$), after linearization and averaging of the coefficients of the equations over the fast variables φ and g, and on the right-hand sides of the equations only with respect to the fast variable φ, take the form

$$\begin{aligned} &A_k\ddot{\psi}_k\sin^2\theta + C_k(\ddot{\varphi}_k + \ddot{\psi}_k\cos\theta)\cos\theta - H_k\dot{\theta}_k\sin\theta + (-1)^k[h\Delta\psi\sin^2\theta + \\ &+ d_2(\Delta\dot{\psi} + \Delta\dot{\varphi}\cos\theta - \dot{\varphi}\sin\theta\Delta\theta)] = -M_k\sin^2\theta\sin2g; \quad k = 1, 2 \end{aligned} \tag{2.7}$$

$$\begin{aligned} &A_k\ddot{\theta}_k + H_k\dot{\psi}_k\sin\theta + (-1)^k[h\Delta\theta + d_2(\Delta\dot{\theta} + \dot{\varphi}\Delta\psi\sin\theta)] = \\ &= -\frac{1}{2}M_k[2\cos2\theta\theta_k + \sin2\theta(1-\cos2g)]; \quad H_k = C_k\dot{\varphi}; \quad k = 1, 2 \end{aligned} \tag{2.8}$$

$$C_k(\ddot{\varphi}_k + \ddot{\psi}_k\cos\theta) + (-1)^k d_2(\Delta\dot{\varphi} + \Delta\dot{\psi}\cos\theta) = 0; \quad k = 1, 2 \tag{2.9}$$

When setting up Eqs. (2.7)–(2.9) we took into account the dissipative function (1.9). The expressions for the coefficient d_2 in Eqs. (2.7)–(2.9) represent the difference in the angular velocities of the mantle and the core of the planet in projections onto the axes of rotation, corresponding to changes in the Euler angles ψ, θ, φ. In projections onto the axes of the system of coordinates $Cz_1z_2z_3$, connected with the planet in the unperturbed motion, the difference in the angular velocities of the mantle and the core can be represented in the form

$$\begin{aligned} &\mathbf{\Omega}_0 = \Gamma_0^{-1}(\Gamma_2\boldsymbol{\omega}_2 - \Gamma_1\boldsymbol{\omega}_1) \cong (\Omega_1\cos\varphi - \Omega_2\sin\varphi)\mathbf{e}_1 + \\ &+ (\Omega_1\sin\varphi + \Omega_2\cos\varphi)\mathbf{e}_2 + \Omega_3\mathbf{e}_3, \quad \varphi = \omega t \\ &\Omega_1 = (\Delta\dot{\theta} + \dot{\varphi}\Delta\psi\sin\theta), \quad \Omega_2 = \Delta\dot{\psi}\sin\theta - \dot{\varphi}\Delta\theta, \quad \Omega_3 = \Delta\dot{\psi}\cos\theta + \Delta\dot{\varphi} \end{aligned} \tag{2.10}$$

Here $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the unit vectors of the system of coordinates $Cz_1z_2z_3$.

The linear system of Eqs. (2.7)–(2.9) contains two cyclic coordinates, φ_1 and φ_2, while the structure of the system is such that one can reduce its order from the twelfth to the eighth, thereby eliminating these coordinates from Eqs. (2.7) and (2.8), using Eq. (2.9). As a result, we obtain a system consisting of Eq. (2.7), in which the second term on the left-hand side is not present and the coefficient of d_2 is represented in the form $\Delta\dot{\psi}\sin\theta - \dot{\varphi}\sin\theta\Delta\theta$, and Eq. (2.8); we will call this System *S*.

We will investigate System *S* within the framework of the approximate theory of gyroscopes, neglecting the second derivatives of the angular variables, which eliminates the fast nutational oscillations with small amplitude from consideration. In this case, System *S* can be represented in the form

$$\begin{aligned} &H_1\dot{\theta}_1 + H_2\dot{\theta}_2 = (M_1 + M_2)\sin\theta\sin2g \\ &\dot{x} - \eta(hy + d_2\dot{y} - d_2\dot{\varphi}x) = (\varsigma_2 - \varsigma_1)\sin\theta\sin2g, \quad x = \Delta\theta \\ &\dot{y} + \eta(hx + d_2\dot{x} + d_2\dot{\varphi}y) - (\varsigma_1\theta_1 - \varsigma_2\theta_2)\cos2\theta = \\ &= \frac{1}{2}(\varsigma_1 - \varsigma_2)\sin2\theta(1 - \cos2g), \quad y = \Delta\psi\sin\theta \end{aligned} \tag{2.11}$$

Here we have introduced the following notation

$$\eta = \frac{1}{H_1} + \frac{1}{H_2}, \quad \varsigma_k = \frac{M_k}{H_k}; \quad k = 1, 2$$

The question of the stability of the trivial solution of the homogeneous system of Eq. (2.11) reduces to investigating the roots of the characteristic equation, which has the form

$$\lambda(c_0\lambda^2 + c_1\lambda + c_2) = 0$$

$$c_0 = 1 + d_2^2\eta^2 > 0, \quad c_1 = d_2\left[\frac{2}{H_1H_2}\dot{\varphi} + 2h\eta^2 + \left(\frac{\varsigma_1}{H_1} + \frac{\varsigma_2}{H_2}\right)\cos 2\theta\right] > 0$$

$$c_2 = (h^2 + d_2^2\dot{\varphi}^2)\eta^2 + h\left(\frac{\varsigma_1}{H_1} + \frac{\varsigma_2}{H_2}\right)\cos 2\theta > 0$$

Hence it follows that one root is equal to zero (a consequence of the first integral $H_1\theta_1 + H_2\theta_2 = \text{const}$), while the two other roots have negative real parts.

The particular solution corresponding to constant perturbation on the right-hand side of Eq. (2.11), can be obtained in the form

$$\dot{\psi}_1 = \dot{\psi}_2 = \dot{\alpha}, \quad \dot{\theta}_1 = \dot{\theta}_2 = 0, \quad \Delta\theta \neq 0, \quad \Delta\psi \neq 0, \quad \dot{\alpha} = -\frac{M_1 + M_2}{H_1 + H_2}\cos\theta$$

As a result, we have

$$\begin{gathered} \theta_1 = QM_2, \quad \theta_2 = -QM_1, \quad x_0 = -Q(M_1 + M_2), \quad y_0 = \frac{d_2\dot{\varphi}}{h}x_0 \\ Q = \frac{\varsigma_2 - \varsigma_1}{2\eta[M_1M_2\cos 2\theta + (M_1 + M_2)(h^2 + d_2^2\dot{\varphi}^2)h^{-1}]}\sin 2\theta \end{gathered} \tag{2.12}$$

The periodic particular solution satisfies equations obtained from system (2.11)

$$\begin{gathered} \dot{x} - \eta(hy + d_2\dot{y} - d_2\dot{\varphi}x) = B_1\sin 2g \\ \dot{y} + \eta(hy + d_2\dot{x} + d_2\dot{\varphi}y + a_3x = sB_1\cos 2g \\ B_1 = (\varsigma_2 - \varsigma_1)\sin\theta, \quad a_3 = \left(\frac{M_1H_2}{H_1} + \frac{M_2H_1}{H_2}\right)\frac{\cos 2\theta}{H_1 + H_2} \\ s = \frac{M_1 + M_2}{2\Omega(H_1 + H_2)}\cos 2\theta + \cos\theta \end{gathered} \tag{2.13}$$

We will seek a solution of system (2.13) in the form

$$x = \Delta\theta = \text{Re}[X\exp(2ig)], \quad y = \Delta\psi\sin\theta = \text{Re}[Y\exp(2ig)]$$

The complex coefficients X and Y satisfy a system of linear algebraic equations, which follow from Eq. (2.13),

$$(a_i + ib_1)X - (a_2 + ib_2)Y = -iB_1, \quad (a_2 + ib_2)X + (a_1 + a_3 + ib_1)Y = sB_1$$

$$a_1 + ib_1 = \eta d_2\omega + 2i\Omega, \quad a_2 + ib_2 = \eta(h + 2i\Omega d_2)$$

As a result, we find

$$\begin{gathered} X = B_1[sa_2 + b_1 + i(sb_2 - a_1 - a_3)]W, \quad Y = B_1[sa_1 - b_2 + i(sb_1 + a_2)]W \\ W = [a_1^2 + a_2^2 + a_1a_3 - b_1^2 - b_2^2 + 2i(a_1b_1 + a_2b_2 + a_3b_1/2)]^{-1} \end{gathered} \tag{2.14}$$

According to relations (2.10) we obtain the vector of the relative angular velocity Ω_0 for the steady motion, described by Eqs. (2.12) and (2.14), where

$$\Omega_1 = \mathrm{Re}\{[2i\Omega X + \omega Y]\exp(2ig)\} + \omega y_0$$
$$\Omega_2 = \mathrm{Re}\{[2i\Omega Y - \omega X]\exp(2ig)\} - \omega x_0, \quad \Omega_3 = 0 \tag{2.15}$$

Since the angular velocity of the mantle relative to the core is small and the systems of coordinates $C_1x_1x_2x_3$ and $C_2y_1y_2y_3$, connected with the mantle and the core, are close to one another, the mutual position can be conveniently described by the Krylov angles α, β, γ, assuming[9]

$$O_1(\alpha)O_2(\beta)O_3(\gamma) = \Gamma_2^{-1}\Gamma_1$$

$$O_2(\beta) = \begin{Vmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{Vmatrix} \Rightarrow \begin{Vmatrix} \dot\alpha + \dot\gamma\sin\beta \\ \dot\beta\cos\alpha - \dot\gamma\sin\alpha\cos\beta \\ \dot\beta\sin\alpha + \dot\gamma\cos\alpha\cos\beta \end{Vmatrix} \cong \begin{Vmatrix} \Omega_1\cos\varphi - \Omega_2\sin\varphi \\ \Omega_1\sin\varphi + \Omega_2\cos\varphi \\ 0 \end{Vmatrix}$$

The last vector equation is obtained taking relation (2.10) into account, and from it we obtain the following equations, apart from small second-order terms inclusive in the Krylov angles and their derivatives,

$$\dot\alpha \cong \Omega_1\cos\varphi - \Omega_2\sin\varphi, \quad \dot\beta \cong \Omega_1\sin\varphi + \Omega_2\cos\varphi, \quad \dot\gamma \cong -\dot\beta\alpha$$

The components of the angular velocity $\dot\alpha$ and $\dot\beta$ are small and are represented by the sum of harmonics with frequencies ω and $\omega \pm 2\,\Omega$, while the component $\dot\gamma$, in addition to the harmonics mentioned above, has a constant component $\langle -\dot\beta\alpha \rangle$, where the angular brackets denote average over time of the quantities in the brackets. Taking relations (2.10), (2.12) and (2.15) into account as well as the smallness of the ratio $2\,\Omega/\omega$, we obtain an approximate estimate for the mean angular velocity in the form

$$\langle\dot\gamma\rangle = -\frac{\omega}{2}(x_0^2 + y_0^2) - \frac{1}{2\omega}[|2i\Omega X + \omega Y|^2 + |2i\Omega Y - \omega Y|^2] \tag{2.16}$$

A similar effect is found in gyroscopic devices on a fixed base.[9] Under steady-state conditions $\Delta\dot\varphi = -\Delta\dot\psi\cos\theta$.

If the core and the mantle of the planet are dynamically similar to one another, this means equality of the ratios $A_1C_1^{-1} = A_2C_2^{-1}$, in which case $Q = 0$ and $B_1 = 0$, and none of the mechanical effects mentioned above are present, i.e. the core and the mantle of the planet move as a single rigid body.

The above analysis of the equations of motion has been based on approximate equations obtained by averaging over the fast variables, and a number of assumptions regarding the terms containing the acceleration, which, naturally, leads to approximate results.

3. Mutual motions of the core and the mantle in the Earth-Moon system

As an example, we will consider the Earth-Moon system, which uses the data in Ref. 3 on the structure of the Earth. We will use the kilogram, the meter and the second as the fundamental units of dimensional quantities and we will take the following numerical values of the quantities required for the calculations:

for the solid core of the Earth

$$a = 1.25 \cdot 10^6, \quad m_1 = 9.82 \cdot 10^{22}, \quad C_1 = 5.87 \cdot 10^{34}, \quad A_1 = 5.86 \cdot 10^{34}, \quad \rho_1 = 1.20 \cdot 10^4$$

for the mantle

$$b = 3.49 \cdot 10^6, \quad m_2 = 417 \cdot 10^{22}, \quad C_2 = 7130 \cdot 10^{34}, \quad A_2 = 7100 \cdot 10^{34}, \quad \rho_2 = 0.46 \cdot 10^4$$

for the liquid spherical layer

$$m_0 = 170 \cdot 10^{22}, \quad C_0 = 908 \cdot 10^{34}, \quad A_0 = 906 \cdot 10^{34}, \quad \rho = 10^4, \quad \xi = b/a = 2.79$$

The radius of the Earth, the angular velocities of the secular rotation and of the rotation of the Moon around the Earth are, respectively

$$r_0 = 6.37 \cdot 10^6, \quad \omega = 7.29 \cdot 10^{-5}, \quad \Omega = 2.67 \cdot 10^{-6}$$

For the angle of inclination of the axis of the Earth to the Moon's orbital plane we take $\theta = 23°$.

As a result of a calculation using the formulae derived in Section 2, we obtain the following numerical values of the quantities

$$m_{10} = 1.64 \cdot 10^{22}, \quad m_{20} = 595 \cdot 10^{22}, \quad m_r = 1.63 \cdot 10^{22}, \quad m_{12} = 9.7 \cdot 10^{22}$$

$$\xi_1 = -0.994, \quad \xi_2 = 2.7 \cdot 10^{-3}, \quad g_1/m_{12} = 1.62 \cdot 10^{-10}, \quad h = 1.87 \cdot 10^{24}$$

$$\omega^2 m_r/m_{12} = 8.9 \cdot 10^{-10}, \quad \Omega^2 m_r/(4m_{12}) = 0.3 \cdot 10^{-12}$$

No reliable data are available at the present time on the viscous properties of the liquid layer of the Earth. We will take as an estimate $\mu = 5 \times 10^7$ Pa s.[3] Hence we obtain the coefficients in Eq. (2.3)

$$d_{10} = 4.3 \cdot 10^{-8}, \quad k_{12} = -1.06 \cdot 10^{-9}$$

while the roots of the characteristic equation are

$$\lambda_+ = 2.53 \cdot 10^{-9} + i7.7 \cdot 10^{-6}, \quad \lambda_- = -4.56 \cdot 10^{-8} + i1.38 \cdot 10^{-4}$$

The root λ_- defines the attenuating natural oscillations of the centre of the core with respect to the mantle in a projection onto the plane Cz_1z_2. This motion can be treated as a displacement of a point in the plane along a spiral, coiled round the origin of coordinates. The period of rotation round the spiral is about 12.5 hours. The second root λ_+, on the other hand, defines the motion along an unwinding spiral with a period of about 9.5 days. This motion represents instability of the equilibrium position and, as was noted above, should lead to a certain steady motion of the centre of the planet core about the mantle due to the occurrence of phase transitions between the liquid layer, the core and the mantle. It is possible to determine the radius of this steady motion using corresponding models of the phase transitions and the temperature distribution inside the Earth. These problems are outside the scope of this paper.

The motion of the centre of the core with respect to the centre of the mantle in a projection onto the Cz_3 axis is described by Eq. (2.5), the coefficients in which are

$$k_3 = 3.24 \cdot 10^{-10}, \quad P = -2.07 \cdot 10^{-10}, \quad P_1 = -2.63 \cdot 10^{-10}, \quad P_3 = 0.154 \cdot 10^{-10}$$

As a result, we obtain forced oscillations about the Cz_3 axis in the form

$$q_3(t) = -0.85 \sin g + 3.2 \cdot 10^{-4} \cos g + 0.059 \sin 3g - 0.72 \cdot 10^{-4} \cos 3g$$

The motion is represented as the sum of two harmonic oscillations, one of which has an amplitude of 85 cm and a period of one month, while the second has an amplitude of 6 cm and a period of a third of a month. To estimate the rotations of the core with respect to the mantle in steady motion, we will use formulae (2.12) and (2.14). The corresponding parameters for the Earth were taken as follows:

$$M_1 = 0.107 \cdot 10^{22}, \quad M_2 = 321 \cdot 10^{22}, \quad H_1 = 4.28 \cdot 10^{30}, \quad H_2 = 5.2 \cdot 10^{33}$$

$$h = 1.87 \cdot 10^{24}, \quad d_2 = 2.58 \cdot 10^{27}, \quad Q = 0.96 \cdot 10^{-28}$$

As a result, we obtain the following steady values of the variables

$$\dot{\alpha} = -5.7 \cdot 10^{-10}, \quad \theta_1 = 3.08 \cdot 10^{-4}(1.06'), \quad \theta_2 = -1.03 \cdot 10^{-7}(-0.02'')$$

$$\Delta\theta = x_0 = -3.08 \cdot 10^{-4}(-1.06'), \quad \Delta\psi \sin\theta = y_0 = -3.1 \cdot 10^{-5}(-6.4'')$$

The mean angular velocity of rotation of the core with respect to the mantle is estimated using formula (2.16) to be $\langle\dot{\gamma}\rangle = -3.5 \times 10^{-12}$. This indicates that, in a year, the core of the planet rotates by an angle of 0.38′ with respect to the mantle.

In addition to constant deviations of the axes, connected with the core of the planet, from the axes connected with its mantle, there are periodic changes in the angles between them at a frequency of $2\,\Omega$, the amplitudes of which are given by formulae (2.14). A calculation of the values of these amplitudes for the numerical values of the parameters given above leads to the following estimates

$$|\Delta\theta| = 7.66 \cdot 10^{-7}(0.037''), \quad |\Delta\psi| = 1.52 \cdot 10^{-5}(0.74'')$$

This means that the angle between the axes of symmetry of the mantle and the core varies periodically with these amplitudes. According to measurements of the angular velocity of rotation of the Earth, harmonics with a period of half a month are found in its spectrum.[3]

The dynamic effects described above may be the reason for seismic activity, together with tidal effects in the deformable mantle and core of the Earth.

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